International Math Olympiad

Shortlisted Problems

41st Mystery Hunt

2021

USA
Cambridge

Contributing Countries

The Organising Committee and the Problem Selection Committee of this IMO thank the following countries for contributing 21 problem proposals:

- Republic of Korea
- United States of America
- United Kingdom
- Japan
- Greece
- Mexico
- Slovenia
- Vietnam
- Spain
- Germany
- Kazakhstan
- Netherlands
- Argentina
- Colombia
- South Africa
- Thailand
- Hong Kong
- Brazil
- Romania
- United Kingdom
- Russian Federation

Algebra

Problem A1 (UNK)
Solve for y in terms of the real number x in the equation

$$(4x+3y)^2 = 4(4x+1)(3y-1).$$

Problem A2 (HEL)
For a real number $x>1$ simplify the expression

$$\sqrt{20x+3+4\sqrt{15x}}+\sqrt{20x+3-4\sqrt{15x}}.$$

Problem A3 (UNK)
A function f from the set of natural numbers to itself satisfies the equation

$$\sum_{d\mid n}[f(d)+1] = 3n$$

for every positive integer n. Given a natural number x, find $f(x)$.

Problem A4 (ESP)
If $x>0$ is a real number, evaluate

$$\sum_{n\geq 0} n\left(1+\frac{1}{\sqrt{x}}\right)^{-n}.$$

Problem A5 (THA)
For each real number $x>0$, determine the smallest natural number d such that a circle of radius $\sqrt{x}$ can be drawn inside a d-dimensional hypercube with side length π.

Problem A6 (VNM)
Find all strictly increasing functions f from the set of real numbers to itself which satisfy $f(24)=5$ and

$$f(x+y)+f(x)+f(y) = 2f(x)f(y)+1$$

for all real numbers x and y.

Combinatorics

Problem C1 (KAZ)

Let n be a positive integer. Let M denote the number of ordered pairs (A, B) of subsets of $\{1, 2, \dots, 2n\}$ such that $A \cap B = \varnothing$ and $|A| + |B| = n$. (The sets A or B could be empty.) Find the largest integer e such that 2^e divides M.

Problem C2 (NLD)

Let n be an integer. Find the largest possible size of a subset of $\{0, 1, 2, \dots, 2n\}$ in which no three elements (not necessarily distinct) have average n.

Problem C3 (ARG)

There is a blackboard with M positive integers written on it. An operation is to choose two of the same number a, a written on the blackboard, and replace them with $a + n$, $a + 2014$. Find the smallest M, in terms of n, for which such an operation could go on infinitely.

Problem C4 (JPN)

Let T be a tournament on n vertices. Its edges and vertices are colored with k colors such that whenever $a \to b$ and $b \to c$ are two edges, then

- the colors of a, b, c, $a \to b$, $b \to c$ are pairwise different;
- the color of a, b, c also differ from the color of the edge between a and c.

Over all tournaments T and all colorings, find the smallest possible value k could take.

Problem C5 (SAF)

Let n be a positive even integer. There are n cards in a row face-down, at least half of which are hearts. Every minute, you can point to two distinct cards. You are happy if they are both hearts; otherwise, nothing happens (in particular neither card is revealed). What is the minimum number of cards you need to point to (i.e. twice the number of minutes) in order to guarantee happiness?

Problem C6 (KOR)

A broken line (polygonal path) consisting of $2n$ line segments has the property that it passes through every point of a $m \times m$ grid. Find the largest possible value of m.

Geometry

Problem G1 (USA)

Let $SNACK$ be a regular pentagon. Let T be the midpoint of minor arc NS of its circumcircle. Point U is chosen in the interior of segment CT such that $TU = TN = TS$. Prove that U, S, A are collinear.

Problem G2 (ROU)

Let ABC be a triangle. Let E be the foot of the altitude from A. Let O and G denote the circumcenter and centroid. Line AG meets BC at D. The circle centered at D with radius DA meets the circumcircle of ABC again at $R \neq A$. The reflection of line OG over line BC meets line AE again at U. Prove that R, O, U are collinear.

Problem G3 (HKG)

Let ABC be an acute scalene triangle with circumcircle ω, orthocenter H, and centroid G. Lines AH and AG meet side BC again at D and E. Let BU and CV be the other two altitudes. Line UV meets line BC at N. The circle with diameter AN meets ω again at M. Line MN meets the circumcircle of $\triangle MDE$ again at L. Let K be the midpoint of AL. Prove that H, K, G are collinear.

Problem G4 (RUS)

Let ABC be a triangle with Feuerbach point P, medial triangle MNL, and intouch triangle DEF. Let $S = FD \cap LM$. Line AB meets the tangent to the incircle at P at a point U. Prove that lines DP, EF, NL are concurrent at a point R such that R, U, S are collinear.

Problem G5 (MEX)

Let ABC be an acute triangle with orthocenter H, circumcenter O, circumcircle Γ, and centroid G. Assume $AB < AC$. Let M, N, P denote the midpoints of AB, AC, BC. The rays MH and NH meet Γ again at points E and D, respectively. Point V lies on line DE such that $\angle VAO = 90°$. Let W denote the reflection of A over OV. Ray WG meets Γ again at L. Point X is chosen on Γ so that

$$\angle LCX + \angle ACB = 90°.$$

Prove that M, E, X are collinear.

Number Theory

Problem NT1 (BRA)

The bank of Rio de Janeiro issues coins of denominations $m < n$, where m and n are positive integers. It turns out there are exactly 4161 positive integers which can't be formed using these coins. Find the ordered pair (m, n) for which $n - m$ is as small as possible.

Problem NT2 (GER)

Find the ordered pair of integers (m, n), not necessarily positive, satisfying

$$58m^2 + 34mn + 5n^2 = 193$$

and for which n is as large as possible.

Problem NT3 (SVN)

Find the ordered pair (m, n) of integers, not necessarily positive, satisfying

$$3^m = (2n - 3)^2 + 8(m - 1)^2$$

and for which $m - n$ is as large as possible.

Problem NT4 (COL)

If π is a permutation on $\{1, 2, \ldots, 9\}$, we define

$$f(\pi) = \pi(1) \cdot \pi(2) \cdot \pi(3) + \pi(4) \cdot \pi(5) \cdot \pi(6) + \pi(7) \cdot \pi(8) \cdot \pi(9).$$

Let m be the smallest integer such that $f(\pi) = 9m$ for some π. Let n be the smallest integer such that $f(\pi) = n$ for some π. Find the ordered pair (m, n).