This summative exam is a milestone on your journey to become mathematicians.

1. A group G is commutative if, for all $g, h \in G, gh = hg$. What is the number of nonisomorphic commutative groups of order 1001?

   Solution: Commutative groups are also known as **abelian groups**. The prime factorization of 1001 is $7 \cdot 11 \cdot 13$. Since this is squarefree, by the classification of finite abelian groups, there is only $\boxed{1}$ abelian group of order 1001, the cyclic group $\mathbb{Z}/1001\mathbb{Z}$.

2. A group of order 105 is guaranteed to have elements of certain orders. Aside from the identity element of order 1, what is the minimum guaranteed order?

   Solution: **Cauchy's theorem** in group theory states that, for any finite group and any prime dividing its order, there exists an element of that prime order. The minimum guaranteed order is thus the smallest prime dividing 105, which is $\boxed{3}$.

3. Assume that, for a positive integer n, $\sqrt[n]{2} = \frac{a}{b}$ is a rational number, with a, b being positive integers. Then $a^n = 2b^n = b^n + b^n$. But it is impossible for an nth power of a positive integer to be the sum of two nth powers of positive integers, giving a contradiction. This argument works for positive integer powers greater than what?

   Solution: **Fermat's Last Theorem** states that there are no positive integer solutions to $a^n + b^n = c^n$ for $n \geq 3$. Thus, the above argument only works for positive integer powers greater than $\boxed{2}$.

4. Compute $\displaystyle\int_0^\pi \frac{3}{\pi(\sqrt{2} - \cos x)}\, dx$. The substitution $u = \tan(x/2)$ may be helpful.

   Solution: Using the **Weierstrass substitution** $u = \tan(x/2)$, we get that $\cos x = \dfrac{1 - u^2}{1 + u^2}$ and $dx = \dfrac{2}{1 + u^2}\, du$. The integral then becomes $\displaystyle\int_0^\infty \frac{6}{\pi(\sqrt{2} - 1) + \pi(\sqrt{2} + 1)u^2}\, du$. Taking the antiderivative and evaluating at the limits gives $\displaystyle \frac{6}{\pi} \tan^{-1}((1 + \sqrt{2})u)\Big|_0^\infty = \frac{6}{\pi}\left(\frac{\pi}{2} - 0\right) = \boxed{3}$.

5. Consider a convex subset of the plane $\mathbb{R}^2$ that is symmetric about the origin. If the area of this subset is greater than 4, then what is the minimum number of lattice points inside the set, other than the origin?

   Solution: By **Minkowski's theorem**, a convex centrally symmetric region in the plane with area greater than 4 contains at least one lattice point x other than the origin. In fact, it must contain at least two, since $-x$ would also be in the region. Thus, when excluding the origin, the minimum number of lattice points is $\boxed{2}$ (which is achieved by a 1.5×3.5 rectangle centered at the origin).

6. Consider the elliptic curve $y^2 = x^3 + 3x^2 + 18x + 4$. For a suitable choice of a, the linear substitution $x \to x - a$ transforms this equation into the more simplified form $y^2 = x^3 + cx + d$. In this form, what is the coefficient c?

   Solution: The substitution $x \to x - 1$ transforms the equation into **Weierstrass normal form**, giving $(x-1)^3 + 3(x-1)^2 + 18(x-1) + 4 = x^3 + 15x - 12$. In this form, $\boxed{c = 15}$.

7. Consider the groups $(\mathbb{Z}/5\mathbb{Z})^\times$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By considering the action of these groups on themselves by left multiplication, we get homomorphisms from these groups to the symmetric group on how many elements?

   Solution: Both of these groups have order 4, and by **Cayley's theorem**, left multiplication gives a homomorphism from these groups to the symmetric group on $\boxed{4}$ elements.

8. Consider the permutation group G of order 20 generated by the permutations (12345) and (1243). G acts transitively on the set $\{1, 2, 3, 4, 5\}$, and only the identity element fixes more than one point. The cyclic subgroup H generated by (1243) fixes the point 5. Let K be the set consisting of the identity and

all elements of G which are not in any conjugates of H. Perhaps surprisingly, K is a normal subgroup of G. What is $|K|$?

Solution: Here, G is a **Frobenius group**.

The permutation (1243) generates a subgroup of order 4 that fixes the number 5. By conjugating by powers of (12345), we can get order-4 subgroups that fix the other four numbers. These five order-4 subgroups only intersect in the identity (since all nonidentity elements move 4 out of 5 numbers, fixing a unique number), giving a total of $5 \cdot (4-1) = 15$ nonidentity conjugate elements. The remaining 5 elements are the identity and the four 5-cycles generated by (12345). These make up the subgroup K (which is called the **Frobenius kernel**), and so $\boxed{|K| = 5}$.

9. Consider the polynomial $f(x) = x^5 - x + 4$ over F_{19}, the field of 19 elements. Let $\varphi : F_{19}[x] \to F_{19}[x]$ be the map that sends $p(x) \mapsto p(x)^{19}$. What is $\varphi(f)(2)$?

   Solution: This map is the **Frobenius endomorphism**, which happens to send every element of the base field F_{19} to itself. This is because $x^{19} = x$ for all $x \in F_{19}$. Therefore, $\varphi(f)(2) = f(2) = 2^5 - 2 + 4 = 34 = \boxed{15} \mod 19$.

10. Consider the quadratic integer ring $R = \mathbb{Z}[\sqrt{-2}]$, with the norm function $N(a + b\sqrt{-2}) = a^2 + 2b^2$. This ring has the property that, for any nonzero elements $x, y \in R$, we can perform a "division with remainder," obtaining elements q and r such that $x = qy + r$, with $r = 0$ or $N(r) < N(y)$. Let $x = 6 + 7\sqrt{-2}$, $y = 2 - \sqrt{-2}$. Among the suitable quotients q and remainders r is a pair with $N(r)$ minimal. What is this minimal norm?

    Solution: As described, the ring R is a **Euclidean domain** with norm function N. We can perform the division algorithm on x and y to get $6 + 7\sqrt{-2} = 3\sqrt{-2}(2 - \sqrt{-2}) + \sqrt{-2}$. The remainder $\sqrt{-2}$ has norm $\boxed{2}$, which must be minimal because $\sqrt{-2}$ is the gcd of x and y.

11. Consider the quadratic integer ring $R = \mathbb{Z}[\sqrt{-5}]$. In this ring, the number 12 can be factored into primes in different ways, for example: $12 = 2 \cdot 2 \cdot 3 = 2(1 + \sqrt{-5})(1 - \sqrt{-5})$. This shows that R does not have unique factorization. However every nonzero proper ideal of R can be factored uniquely as the product of prime ideals. What is the number M of prime ideals in the factorization of the principal ideal $\langle 12 \rangle$, counting multiplicity?

    Solution: A ring like R where every nonzero idea can be factored uniquely into prime ideals is called a **Dedekind domain**. In this ring, the ideal $\langle 12 \rangle$ can be factorized as $\langle 2, 1 - \sqrt{-5}\rangle^4 \langle 3, 1 - \sqrt{-5}\rangle \langle 3, 1 + \sqrt{-5}\rangle$. Thus, there are $\boxed{M = 6}$ prime ideal factors.

12. Consider the subring of complex numbers whose real and imaginary parts are both integers. If $a + bi$ is the greatest common divisor of $12 + 9i$ and $-15 + 20i$, where both a and b are positive, what is b?

    Solution: This ring is known as the **Gaussian integers**. The gcd of $12 + 9i$ and $-15 + 20i$ is $4 + 3i$, so $\boxed{b = 3}$.

13. For an odd prime p and an integer a not divisible by p, the expression $\left(\frac{a}{p}\right)$ is defined to be 1 if there exists an integer m such that $m^2 \equiv a \mod p$, and -1 otherwise. What is $\left(\frac{2}{71}\right)$?

    Solution: $\left(\frac{2}{71}\right)$ is an example of a **Legendre symbol**. By a supplement to the Law of Quadratic Reciprocity, since $71 \equiv -1 \mod 8$, 2 is indeed a quadratic residue mod 71, so the value of the symbol is equal to $\boxed{1}$.

14. If S and T are subsets of $\mathbb{R}^2$, define their sum $S + T = \{s + t | s \in S, t \in T\}$. Assume that T is the right triangle with vertices $(0,0), (1/2, 0), (0, 1/2)$, that S is a square with its bottom left vertex at the origin with its sides parallel to the coordinate axes, and that the area of $S + T$ is $\frac{17}{8}$. What is the area A of the square?

Solution: The sum defined above is the **Minkowski sum**. If the square has side length x, then $S+T$ will be the convex hull of the points $(0,0), (x+1/2,0), (x+1/2,x), (x,x+1/2), (0,x+1/2)$. This polygon is a square with its top-right corner cut off, so its area is equal to $(x+1/2)^2-(1/2)(1/2)^2 = x^2+x+1/8$. We want this expression to equal $17/8$, so $x^2+x=2$ and $x=1$. Thus for the area of the square, we get $\boxed{A=1}$.

15. If a convex polyhedron has 8 faces and 7 vertices, what is E, its number of edges?

    Solution: By **Euler's polyhedron formula**, $V-E+F = 7-E+8 = 2$, so $\boxed{E=13}$.

16. In one of the main ways of constructing $\mathbb{R}$ from $\mathbb{Q}$, the irrational number $2\sqrt{2}$ can be defined as the set $\{x \in \mathbb{Q} | x^2 < \alpha \text{ or } x < 0\}$. What rational number α is this upper bound?

    Solution: This is the **Dedekind cut** definition for $2\sqrt{2} = \sqrt{8}$, so the upper bound is $\boxed{8}$.

17. Let γ be the unit circle in the complex plane, oriented counterclockwise. Compute the integral $F = \int_\gamma \frac{\cos z}{i\pi z}\, dz$.

    Solution: By **Cauchy's integral formula**, $\int_\gamma \frac{\cos z}{i\pi z}\, dz = 2\cos 0$, so $\boxed{F=2}$.

18. Let G be a group of order 49, and let H be a nontrivial proper subgroup of G. What is $|H|$?

    Solution: By **Lagrange's theorem**, the order of a subgroup must divide the order of the group, so the order of H must be one of $1, 7, 49$. Thus, for H to be a nontrivial proper subgroup, we must have $\boxed{|H|=7}$.

19. Let H be the dihedral group of a polygon with an odd number of sides. Then, $[H,H]$ is the subgroup generated by commutators $hkh^{-1}k^{-1}$, with $h,k \in H$. It turns out that the commutator subgroup is normal. If we consider the quotient group $G = H/[H,H]$, then what is $|G|$?

    Solution: The group G is the **abelianization** of H, and it is the largest abelian quotient of H. Equivalently, the commutator subgroup is the smallest normal subgroup that gives an abelian quotient. If n is the number of sides of the polygon, then we have the group presentation $H = \langle r, s | r^n = s^2 = 1, srs = r^{-1}\rangle$. With this, the commutator $rsr^{-1}s^{-1}$ is equal to r^2, and since n is odd, $\langle r^2\rangle = \langle r\rangle$. Since $\langle r\rangle$ is a normal subgroup of index 2, it results in an abelian quotient of $\mathbb{Z}/2\mathbb{Z}$. Therefore, $[H,H] = \langle r\rangle$, and $\boxed{|G|=2}$.

20. Let j be the highest power such that $2^j | 20!$. What is j?

    Solution: By **Legendre's formula**, the largest power of 2 dividing 20! is $\lfloor \frac{20}{2}\rfloor + \lfloor \frac{20}{4}\rfloor + \lfloor \frac{20}{8}\rfloor + \lfloor \frac{20}{16}\rfloor = 10+5+2+1 = 18$. Thus, $\boxed{j=18}$.

21. The content of a polynomial in $\mathbb{Z}[x]$ is the greatest common divisor of its coefficients. If f has content 3 and g has content 2, then what is the content of their product fg?

    Solution: By **Gauss's lemma** for polynomials, the content of a product is the product of the individual contents, so the content of fg is $2 \cdot 3 = \boxed{6}$.

22. The series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ converges for complex numbers s with real part greater than 1. This resulting function has an analytic continuation to all complex numbers except $s=1$. With ζ continued in this way, what is $L = \frac{1}{\zeta(-1)}$?

    Solution: Note that ζ is the **Riemann zeta function**. We have that $\zeta(-1) = -1/12$, so $\boxed{L=-12}$.

23. There is a unique quadratic polynomial that passes through the points $(1,7), (-3,35)$, and $(5,43)$. What is this polynomial's constant term?

Using **Lagrange interpolation**, we can compute the interpolating polynomial as $p(x) = 7\left(\frac{x-(-3)}{1-(-3)}\right)\left(\frac{x-5}{1-5}\right) + 35\left(\frac{x-1}{-3-1}\right)\left(\frac{x-5}{-3-5}\right) + 43\left(\frac{x-1}{5-1}\right)\left(\frac{x-(-3)}{5-(-3)}\right) = 2x^2 - 3x + 8$. The constant term of this polynomial is $\boxed{8}$.

24. We can approximate the area under the curve $y = x^2$ from $x = 1$ to 3 by splitting the interval $[1, 3]$ in half and drawing a rectangle from the x-axis to the point on the curve above the rightmost endpoint of each subinterval. What is the total area of these two rectangles?

    Solution: For this **Riemann sum**, the two rectangles both have width 1, and heights 4 and 9, so the total area is $\boxed{13}$.

25. What is $e^{\pi i}$?

    Solution: By **Euler's formula**, $e^{\pi i} = \cos\pi + i\sin\pi = \boxed{-1}$.

26. What is N, the number of nonisomorphic trees on 4 labeled vertices?

    Solution: By **Cayley's formula**, the number of such trees is 4^{4-2}, so $\boxed{\text{N=16}}$.

27. What is the distance d between the points $(1/2, 1/2, 1/2, 1/2)$ and $(0, 1, 1, 0)$ in $\mathbb{R}^4$?

    Solution: The **Euclidean distance** between these points is $\sqrt{(1/2-0)^2 + (1/2-1)^2 + (1/2-1)^2 + (1/2-0)^2}$, and so $\boxed{d = 1}$.

28. What is the smallest positive integer i such that $x^i \equiv 1 \bmod 13$ for all integers x not divisible by 13?

    Solution: By **Fermat's little theorem**, $\boxed{i = 13 - 1 = 12}$.