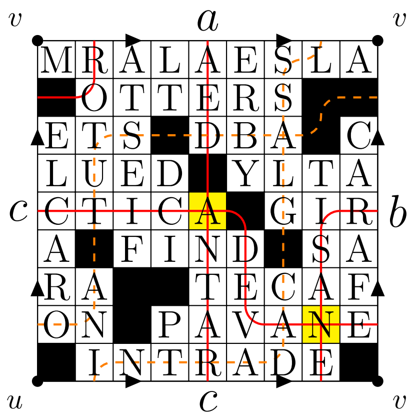
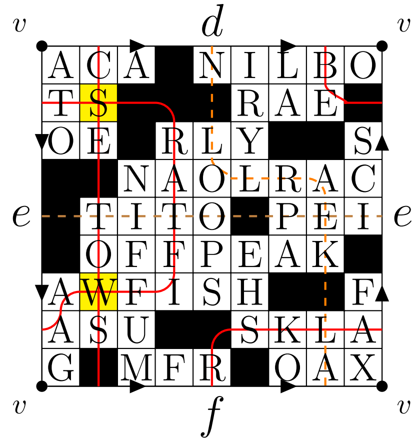
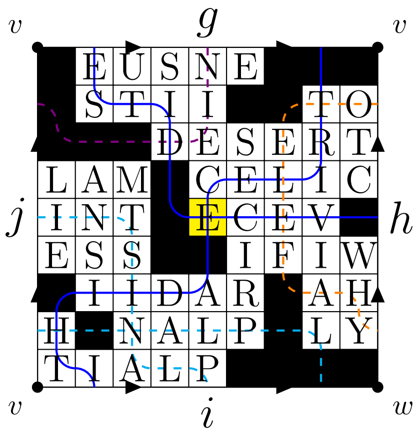
$F_1$:



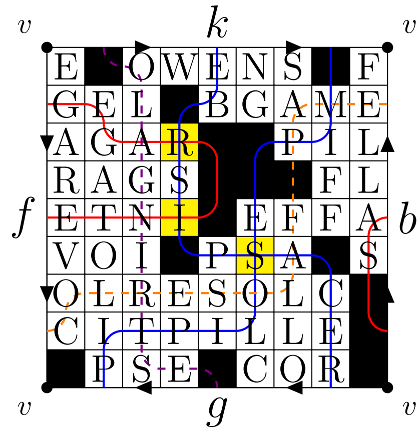

$F_2$:



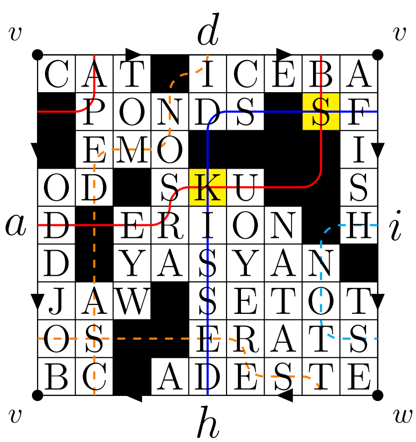

$F_3$:



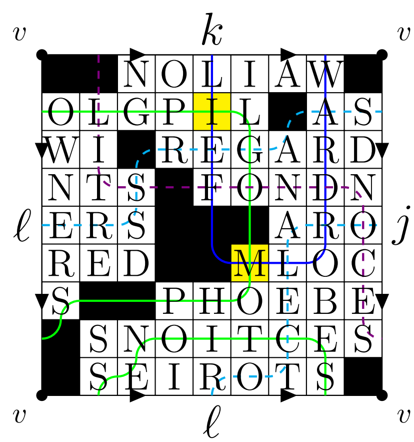

$F_4$:




$F_5$:




$F_6$:




The completed crossword grid forms a CW complex, $X$, with three vertices, $u, v$ and $w$, twelve edges, $a, \ldots, \ell$, and six faces $F_1, \ldots, F_6$. The paths drawn in the original grid join up to form seven loops in $X$, colored in red, blue, green, orange, purple, cyan and brown.

The unclued entries give the message FIND TRIVIAL LOOPS' INTERSECTIONS. This suggests determining which of these loops have trivial homotopy type in $X$, that is, which loops can be continuously deformed to a point. It is easy to see that the red, blue and green loops can all be deformed to a point. Moreover, it is easy to convince yourself that the same is not true for the other four loops (a proof of this fact is given below). Taking the letters at the intersection points (including self-intersects) between the red, blue and green loops spells out the message ANSWER IS SKIM.

---

We now prove that the orange, purple, cyan and brown loops have nontrivial homotopy type. This will require some basic results from algebraic topology[1].

We have surjective maps

$$\pi_1(X) \twoheadrightarrow H_1(X;\mathbb{Z}) \twoheadrightarrow H_1(X;\mathbb{Z}_2)$$

where the first follows from the Hurewicz theorem and the second is from the universal coefficient theorem. It will thus suffice to show that these loops are nontrivial in $H_1(X;\mathbb{Z}_2)$.

Clearly the space $X$ is a closed 2-manifold. From the above CW complex, its Euler characteristic is $\chi(X) = 3 - 12 + 6 = -3$. By the classification of surfaces this implies that $X$ is homeomorphic to $N_5$, the nonorientable surface of genus 5. Hence

$$H_1(X;\mathbb{Z}_2) \cong H_1(N_5;\mathbb{Z}_2) \cong \mathbb{Z}_2^5.$$

Now we compute $H_1(X;\mathbb{Z}_2)$ via cellular homology, by taking the cell complex structure above. The cellular chain complex of $X$ is:

$$0 \to \mathbb{Z}_2 F_1 \oplus \cdots \oplus \mathbb{Z}_2 F_6 \xrightarrow{\partial_2} \mathbb{Z}_2 a \oplus \cdots \mathbb{Z}_2 \ell \xrightarrow{\partial_1} \mathbb{Z}_2 u \oplus \mathbb{Z}_2 v \oplus \mathbb{Z}_2 w \to 0.$$

Computing boundaries, we see that

$$\begin{aligned}
&\partial_1(a) = \partial_1(b) = \partial_1(d) = \partial_1(e) = \partial_1(f) = \partial_1(g) = \partial_1(j) = \partial_1(k) = \partial_1(\ell) = v + v = 0 \\
&\partial_1(c) = u + v \\
&\partial_1(h) = \partial_1(i) = v + w.
\end{aligned}$$

From this we see that $\ker \partial_1$ has the basis $\{a, b, d, e, f, g, j, k, \ell, h + i\}$ and consequently $H_1(X;\mathbb{Z}_2)$ is spanned as a $\mathbb{Z}_2$-vector space by $\{[a], [b], [d], [e], [f], [g], [j], [k], [\ell], [h + i]\}$.

[1] References for all results used here may by found in the book *Algebraic Topology* by Hatcher (available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`).

Now we have

$$\begin{aligned}
\partial_1(F_1) &= a+b+c+c = a+b\\
\partial_1(F_2) &= d+e+f+e = d+f\\
\partial_1(F_3) &= g+h+i+j = g+j+(h+i)\\
\partial_1(F_4) &= k+b+f+g\\
\partial_1(F_5) &= d+i+h+a = a+d+(h+i)\\
\partial_1(F_6) &= k+j+\ell+\ell = k+j.
\end{aligned}$$

This gives the relations

$$\begin{gathered}
[a]+[b]=0 \Rightarrow [b]=[a]\\
[d]+[f]=0 \Rightarrow [f]=[d]\\
[g]+[j]+[h+i]=0 \Rightarrow [h+i]=[g]+[j]\\
[k]+[b]+[f]+[g]=0 \Rightarrow [g]=[b]+[f]+[k]\\
[a]+[d]+[h+i]=0 \Rightarrow [h+i]=[a]+[d]\\
[k]+[j]=0 \Rightarrow [k]=[j].
\end{gathered}$$

which give

$$\begin{aligned}
[b] &= [a]\\
[f] &= [d]\\
[g] &= [a]+[d]+[j]\\
[k] &= [j]\\
[h+i] &= [a]+[d].
\end{aligned}$$

Thus $H_1(X;\mathbb{Z}_2)$ is spanned by $\{[a],[d],[e],[j],[\ell]\}$. But as $\dim_{\mathbb{Z}_2} H_1(X;\mathbb{Z}_2)=5$, these classes must form a basis for $H_1(X;\mathbb{Z}_2)$. Now homotoping the loops orange, purple, cyan and brown so that they are *cellular* maps $S^1 \to X$, we get that

$$\begin{aligned}
[\text{orange}] &= [a]+[k]+[e]+[a]+[b] = 3[a]+[e]+[j] = [a]+[e]+[j] \neq 0\\
[\text{purple}] &= [b]+[j] = [a]+[j] \neq 0\\
[\text{cyan}] &= [k]+[\ell]+[j] = 2[j]+[\ell] = [\ell] \neq 0\\
[\text{brown}] &= [d]+[e] \neq 0.
\end{aligned}$$

So indeed all four of these loops are nontrivial in $H_1(X;\mathbb{Z}_2)$, and thus in $\pi_1(X)$.